# CALCULUS-HELP.COM'S

# 2003

## Superbowl of High School Calculus

### Final Reminders and Last Requests:

1. Members of the same group may work together however they wish, but separate groups may not work together.
2. You may not use a calculator, computer, abacus, pile of smooth stones, or any other computing device of any type or level of technology.
3. Show all your work for every step of every problem, and be as neat as possible. Organization and presentation may serve as a tie-breaker to comparable solutions! Attach additional sheets as necessary.
4. Use a dark pencil or pen. You're faxing your answers, and I need to be able to read them!
5. Please simplify all answers as much as possible; that means you need to evaluate any trigonometric functions on the unit circle. (For example, don't leave "$\cos\pi$" as a solution. The answer would be "–1.")
6. Unless explicitly stated otherwise, assume all angles are expressed as radians.
7. If one part of a problem refers to a function, like $f(x)$, you can assume that any other references to $f(x)$ in the other parts of the same problem refer to the same function. However, if you see that notation in a completely different problem, it probably refers to a completely different function.
8. At 12 noon EDT, it's all over but the crying (unless you have an authorized extension).
9. The test administrator (teacher or staff member) should sign the bottom of this form when the test is over to indicate that the participants worked only in groups of 3, got no additional help, and did not use restricted technology, books, or notes.
10. **Submit only one solution per problem per team!**

**Team Name:** ______________________ **School Name:** _________________________

**Team Members:** ______________________________________________________________

**Test administrator's signature:** ______________________________________________

# All the Legal Stuff



**Note that the rubric attached to each question represents only a *guideline* for grading; the judge's word is final on all grades, even if you disagree. Rule 3 on the release sheet says to show work for every step of every problem, and if you didn't, you probably lost points!**

## Question 1: You Can't Spell "Fundamental" Without "Fun"!

If the even function $f(x)$ and the odd function $g(x)$ are continuous for all $x$, and you know that $\int_a^b f(x)\,dx = 12$ and $\int_a^b g(x)\,dx = -5$, evaluate the following:

a) $\int_b^a \left(f(x)+g(x)\right) dx$

b) $\int_a^a \left(f(x)-g(x)\right) dx$

c) $\int_a^b \left(g(x)+3\right) dx$

d) $\int_{2a}^{a+b} f(x-a)\,dx$

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### Solutions

| | | |
|---|---|---|
| (a) | $-7$ | (2 points, 1 for negative sign) |
| (b) | $0$ | (2 points) |
| (c) | $-5 + 3(b - a)$ | (3 points, only +1 if no variables) |
| (d) | $12$ | (3 points, 1 for sign) |

## Question 2: This Table's Firing Blanks

Assume that the functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ are continuous and differentiable everywhere, as are their derivatives. In the below table, you'll find all of those functions evaluated at some location $x = a$. However, some of the values have been omitted.

| | $a$ |
|---|---|
| $f(x)$ | 1 |
| $f'(x)$ | 2 |
| $g(x)$ | |
| $g'(x)$ | |
| $h(x)$ | $-\frac{1}{3}$ |
| $h'(x)$ | |
| $j(x)$ | |
| $j'(x)$ | 0 |

If $h(x) = \dfrac{f(x)}{g(x)}$ and $j(x) = f(x) \cdot g(x)$, respond to the following questions:

a) Fill in all the missing table values, showing the work that leads to your answers.

b) Evaluate $p'(a)$ if $p(x) = f(x) \cdot g(x) \cdot j(x)$.

## Solutions

(a)

| | $a$ |
|---|---|
| $f(x)$ | 1 |
| $f'(x)$ | 2 |
| $g(x)$ | **–3** |
| $g'(x)$ | **6** |
| $h(x)$ | $-\frac{1}{3}$ |
| $h'(x)$ | $-\frac{\mathbf{4}}{\mathbf{3}}$ |
| $j(x)$ | **–3** |
| $j'(x)$ | 0 |

(2 points each, 1 off for incorrect sign)

(b) 0

(2 points, all or nothing)

## Question 3: Awards for Best Theorem in a Supporting Role

Do you know who won the Oscar last year for *Best Actor in a Leading Role*? I do; it was Denzel Washington. Here's a tougher question: Who won the award for Best Actor in a Supporting Role? Does the name Jim Broadbent ring a bell? I have absolutely no idea who he is, and compared to Denzel, frankly, he's small potatoes, Oscar or not.

Some calculus theorems have lives as tough as good ol' Jimmy B. Let's face it. The Fundamental Theorem of Calculus gets all the attention, leaving other important, but less famous theorems, feeling lonely and neglected. My solution to this dilemma is the following set of nominations for Best Supporting Theorem in the Field of Calculus, and what nomination would be complete without a problem to go with each?

a) **Nominee One: The Mean Value Theorem**
Given some continuous and differentiable function $g(x)$ with a $y$-intercept of 4, and possessing the additional characteristic that $2 \leq g'(x) \leq 5$ for all $x$, use the Mean Value Theorem to find the largest possible value of $g(6)$.

b) **Nominee Two: The Intermediate Value Theorem**
Use the Intermediate Value Theorem to prove that a real number exists which is *exactly* three less than its own cube.

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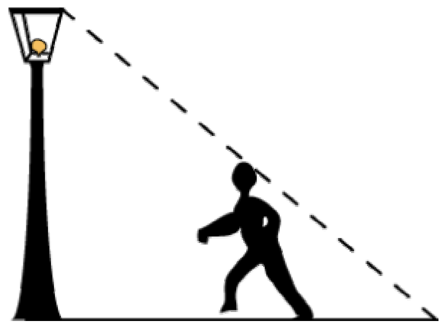
## Solutions

| | | |
|---|---|---|
| (a) | 1 point: | Mean Value Theorem setup |
| | 2 points: | $\frac{g(6)-g(0)}{6}=5$ |
| | 2 points: | Solution of 34 |
| (b) | 2 points: | $x = x^3 - 3$ |
| | 3 points: | Show $x$-intercepts bounding irrational intercept ($x$ = 1, 2 or better) |

## Question 4: I'm Being Followed by a Moon Shadow

Assuming that I am 5 feet, 10 inches tall and am walking toward a lamppost at a constant rate of 3 feet per second, how fast is the tip of my shadow moving?



## Solution

In the solution, $x$ = distance from person to pole, $y$ is distance from shadow tip to pole.

1 point: Realize that height of lamppost is constant, set variable ($L$)
1 point: Recognize similar right triangles
2 points: Ratio of big triangle to small is $L : \frac{35}{6}$ or $\frac{6L}{35}$
2 points: Proportion equation such as $\frac{6L}{35} = \frac{y}{y-x}$

$$6Ly - 6Lx = 35y$$

$$6L\frac{dy}{dt} - 6L\frac{dx}{dt} = 35\frac{dy}{dt}$$

2 point: $dx/dt$ is –3 (negative because decreasing length of $x$)

$$\frac{dy}{dt}(6L - 35) = 6L(-3)$$

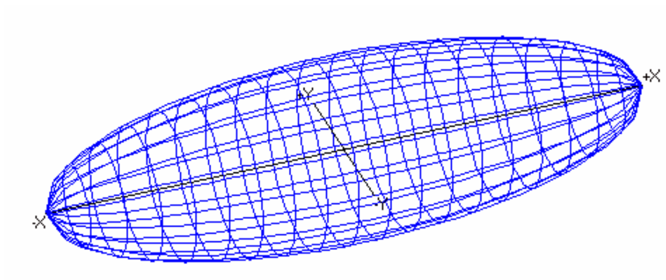
2 points: Solution: $\frac{dy}{dt} = \frac{-18L}{6L-35}$

## Question 5: Read My Ellipse

A two-dimensional region $A$ is defined by an ellipse centered at the origin, possessing a horizontal major axis of length 10 and a minor axis of length 3.

a) Give the two functions, $f(x)$ and $g(x)$, which respectively define the upper and lower curves that bound the ellipse.

b) Calculate the volume of the solid generated by rotating $f(x)$ about the $x$-axis.



c) Assume that $A$ is the base of a solid whose cross-sections are semi-circles. Calculate the volume of that solid using the simplest method possible, justifying the method you use.

## Solutions

| | | |
|---|---|---|
| (a) | 3 points: | $f(x)=\frac{\sqrt{225-9x^2}}{10},\ g(x)=-\frac{\sqrt{225-9x^2}}{10}$ |
| (b) | 2 points: | Disc Method: $\pi\int_a^b (r(x))^2\,dx$ |
| | 2 points: | Correct answer: $15\pi$ |
| (c) | 1 point: | Half of answer from (b) |
| | 2 points: | Disc method uses full discs. Semicircles give exactly half volume. |

## Question 6: One of These Things Is Not Like The Other…

If Mary-Kate and Ashley Olsen have taught us anything, it's that even though things may look the same, they can, in fact, be very different. On the surface, they appear *exactly* the same, except one of them loves Dave Coulier (Uncle Joey from *Full House*) and the other secretly despises him. But, I digress.

All of these integrals look similar, but when you integrate them, you'll find they require completely different methods. (Just like you learn, upon further investigation, that the Olsen's are actually *fraternal* twins. Chilling!)

(a) $f(x) = \int \frac{x+1}{x}\, dx$

(b) $g(x) = \int \frac{x}{x+1}\, dx$

(c) $h(x) = \int \frac{x}{x^2+1}\, dx$

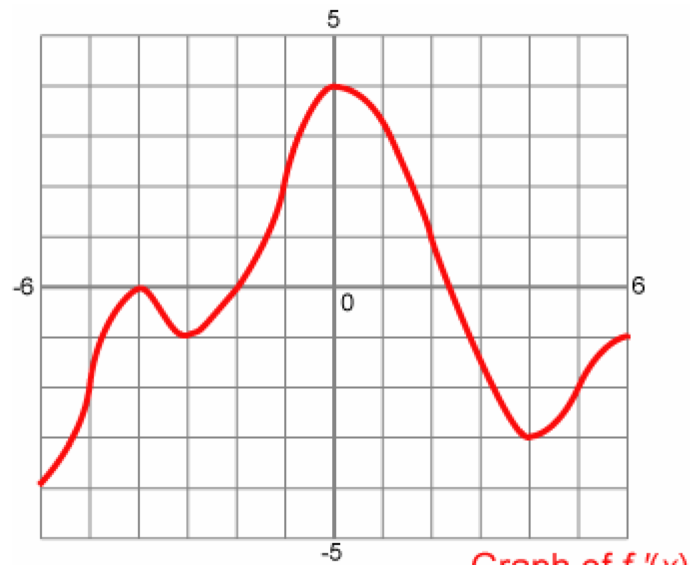
(d) $j(x) = \int \frac{1}{x^2+1}\, dx$

## Solutions:

| | | |
|---|---|---|
| | 2 points | constant of integration (+ $C$ ), 1/2 point each with correct answer |
| (a) | 2 points | $x + \ln\|x\| + C$ |
| (b) | 2 points | $x - \ln\|x + 1\| + C$ |
| (c) | 2 points | $\frac{1}{2}\ln\left(x^2+1\right)+C$ |
| (d) | 2 points | arctan $x + C$ |

## Question 7: Analyze This

Below, I have drawn the graph of $f'(x)$, the derivative of some function $f(x)$.



Graph of $f'(x)$

a) At what value(s) of $x$ does $f(x)$ have a relative extrema point? Classify each as either a relative maximum or minimum.

b) Assuming $f''(x)$ is continuous, how many roots does it have on the $x$-interval [–5,5]?

c) If $g(x)=\int_{-3}^{x} f(t)\,dt$, rank the following function values from *least* to *greatest*: $g(-6)$, $g(-4)$, $g(-2)$, $g\left(-\frac{1}{2}\right)$.

d) Assuming that the point (4,–3) is a relative minimum on the above graph, rank the following values from *least* to *greatest*: $f'(4)$, $f''(4)$, $f'''(4)$.

e) Approximate the average value of $f'(x)$ on the $x$-interval [-6,6], and show the analysis that leads to your answer.

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**Solutions:** (2 points each)

**(a)** $x=-2$ (min), $x\approx\frac{7}{3}$ (max)

**(b)** 4: number of relative extrema points

**(c)** $g(-2), g(-4), g\left(-\frac{1}{2}\right), g(-6)$

**(d)** $f'(4), f''(4), f'''(4)$

**(e)** $\frac{1}{12}\int_{-6}^{6} f'(x)\,dx\approx -.1$

*(Answers to part (e) can vary a bit)*

## Question 8: Absolutely Horrifying

The function $f(x) = \left| \frac{2}{3}x^3 - x^2 - \frac{11}{3}x + 2 \right|$ is not only attractive, but has a fantastic personality as well. In case you're not convinced, here are four questions about $f$ to help you get to know one another better:

a) Sketch the graph of $f(x)$ on the interval $-4 \le x \le 4$.

b) Design a piecewise-defined function $g(x)$ such that $g(x) = f'(x)$. In case you don't know, a piecewise-defined function gives different outputs based on defined input intervals and is usually written like this:

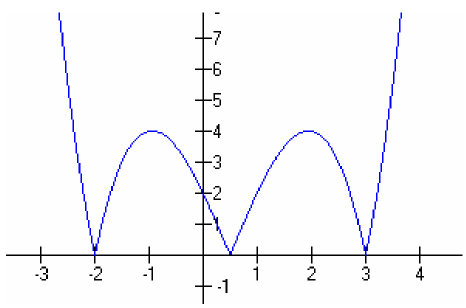
$$g(x) = \begin{cases} m(x), & x \le a \\ n(x), & x > a \end{cases}$$

c) Evaluate $\int_0^4 f(x)\,dx$.

d) Discuss the differentiability and continuity of $f$ and $g$.

**Solutions:**

(a)




3 points: 1 point each root: $-2, \frac{1}{2}, 3$

1 point: Graph all positive with points

(b) $$g(x) = \begin{cases} 2x^2 - 2x - \frac{11}{3}, & \left\{-2 < x < -\frac{1}{2}\right\} \cup \{x > 3\} \\ -2x^2 + 2x + \frac{11}{3}, & \{x < -2\} \cup \left\{\frac{1}{2} < x < 3\right\} \end{cases}$$

1 point: Correct derivative
1 point: Derivative opposites
1 point: Intervals

(c) 1 point: Split into three integrals: $\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 3\right), (3, 4)$

1 point: Answer: $\frac{49}{96} + \frac{625}{96} + 6 = \frac{625}{48}$

(d) 1 point: $f$ not differentiable, $g$ not continuous or differentiable at roots of $f$

## Question 9: What an Orc Wants

(Based on the book *The Lord of the Rings: The Two Towers*, by J.R.R. Tolkien) The Battle at Helm's Deep is about to begin. Unfortunately for humankind, it looks like Gandalf isn't going to be around when the first blows are struck. Gimli the dwarf and Legolas the elf, despite staring straight into the face of certain doom (in the form of thousands of marauding orcs), nonetheless maintain a competitive relationship. Throughout the conflict, they will call out the total number of orcs they have dispatched since the battle begun, and I have collected that data in the table below.

| Minutes Into Battle | Total Number of Gimli's Victims | Total Number of Legolas's Victims |
|---|---|---|
| 0 | 0 | 0 |
| 15 | 0 | 7 |
| 30 | 1 | 22 |
| 45 | 8 | 30 |
| 60 | 15 | 33 |
| 75 | 24 | 42 |
| 90 | 37 | 44 |
| 105 | 49 | 52 |
| 120 | 58 | 57 |

a) If $g(t)$ is a mathematical model of the cumulative number of Gimli's victims at any time $t$ minutes into the battle, approximate the area between $g(t)$ and the $x$-axis on the $t$ interval [0,120] using right-hand Riemann sums with 8 rectangles.

b) If $l(t)$ is a mathematical model of the cumulative number of Legolas's victims at any time $t$ minutes into the battle, approximate the area between $l(t)$ and the $x$-axis on the $t$ interval [0,120] using midpoint Riemann sums with 4 rectangles.

c) **True or False:** *Since a definite integral represents total change, the answers to questions (a) and (b) above approximate the total number of each fighter's victims.* Justify your answer.

d) **True or False:** $l''(t) > 0$ *on [0,30]*. Justify your answer.

e) Find an interval $[a,b]$ on $g(t)$ and an interval $[c,d]$ on $l(t)$ for which the average rate of change of orc casualties are equal.

### Solutions:

(a) 15 (0 + 1 + 8 + 15 + 24 + 37 + 49 + 58) = 2880 square units (2 points)

(b) 30 (7 + 30 + 42 + 52) = 3930 square units (2 points)

(c) False: $l$ and $g$ alone give such an approximation (2 points)

(d) $l$ increasing at an increasing rate this concave up and $l'' > 0$ (2 points)

(e) Answers vary (2 points all or nothing)

## Question 10: If You Could Read My Mind

I am thinking of four functions: $f(x)$, $g(x)$, $h(x)$, and $k(x)$. Here are some hints that should help you determine what, exactly, those functions are:

- $f(x)$ and $g(x)$ are trigonometric functions with identical roots
- $h\left(k\left(x\right)\right)=k\left(h\left(x\right)\right)=x$
- $h(x)$ and $g(x)$ share a vertical asymptote
- $k^{(n)}\left(x\right)=k\left(x\right)$, in other words $\dfrac{d^n y}{dx^n}\left(k\left(x\right)\right)=k\left(x\right)$, for all integers $n$, if $n\geq 1$

Based on those clues, do the following:

a) Correctly identify the functions I named $f(x)$, $g(x)$, $h(x)$, and $k(x)$, making sure to tell me which is which.

b) If $m\left(x\right)=k\left(g\left(x\right)\right)$ and $p\left(x\right)=\dfrac{f\left(x\right)}{h\left(x\right)}$, find $m'\left(x\right)$ and $p'\left(x\right)$.

### Solutions:

(a) 2 points each $f(x) = \cos x$
$g(x) = \cot x$
$h(x) = \ln x$
$k(x) = e^x$

(b) 1 point each:

$$m'\left(x\right)=-\csc^2 x\cdot e^{\cot x}$$

$$p'\left(x\right)=\frac{-\ln x\sin x-\dfrac{1}{x}\cos x}{\left(\ln x\right)^2}$$

*Note:* First bullet during Superbowl competition omitted the phrase "with identical roots," so $f(x)$ and $p'(x)$ were not graded. Furthermore, these answers were accepted:

$g(x) = \csc x$
$m'(x) = -\csc x \cot x\, e^{\csc x}$

# Statistical Analysis for 2003 Superbowl Questions

| Question | Mean | Median | Mode | Standard Deviation |
|---|---|---|---|---|
| **1. Fundamental Theorem** | 7.26 | 7 | 10 | 2.57 |
| **2. Table's Firing Blanks** | 8.57 | 10 | 10 | 2.26 |
| **3. Supporting Theorems** | 3.29 | 2 | 0 | 3.53 |
| **4. Moon Shadow** | 2.32 | 0 | 0 | 3.52 |
| **5. Read My Ellipse** | 2.95 | 2 | 0 | 3.28 |
| **6. Not Like the Other...** | 6.19 | 7 | 9 | 3.02 |
| **7. Analyze This** | 5.63 | 6 | 8 | 2.76 |
| **8. Absolutely Horrifying!** | 3.21 | 2.5 | 0 | 2.79 |
| **9. What an Orc Wants** | 4.71 | 5 | 0 | 3.39 |
| **10. Read My Mind** | 4.03 | 6 | 7 | 3.11 |
| **Total Score** | 48.17 | 47.5 | 30.5 | 21.02 |